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Simple Games on Networks

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Abstract

In the same way that traditional game theory captured the minds of economists and allowed complex problems to be studied using simple models, so have games on networks come to be used in computer science. In particular, previous work has focused on extending a two-player *base game* M over a network G by having each vertex in G chose a strategy from the base game and play it simultaneously against all adjacent nodes. Similarly, the utility for each vertex becomes the sum of the utility of that node in each of the games it plays against its neighbors. The resulting *networked game* is called $M \oplus G$. This paper seeks to augment the body of existing work by studying a few similar networked games and finding key characteristics like the existence of Nash equilibria, the price of anarchy, the price of stability, the convergence speed of best response dynamic, and the difficulty of finding the optimal solution. We also reproduce the result that the class of exact potential games is isomorphic to the class of congestion games with a proof that is drastically more readable than the original.

1 Introduction

As the internet has grown, so has our fascination with networks in general. The interactions that occur within social networks seem especially compelling, as such networks represent the first time picture of human interaction that seems relatively complete and can be studied rigorously. To model these interactions, scientists have turned to game theory, making the class of *graphical games* [7] a natural target for research. Here, we consider several different games on networks. Originally defined and analyzed in [3], the Matching Game serves as a useful jumping off point for considering a new game where players receive utility by playing a strategy that is distinct from their opponents. Known as the Mismatching Game, this paper studies many core facets of the game including the existence of Nash equilibria, the price of anarchy [8], the price of stability [1], the computability of the optimal solution, and the convergence of best response dynamics. We also introduce a second new game, the Affiliation Game, which is a hybrid of the Matching and Mismatching games and a natural extension of correlation clustering [4] [2] to game theory. This model also generalizes models of the *party affiliation* game introduced by Fabrikant et.al [5] and Balcan et. al. [1]

Finally, when attempting to analyze the convergence speed of best response dynamics, we came across the complexity class PLS [6] [11]. Existing work introduced the class of congestion games [10] and proved that every potential game is isomorphic to a congestion game [9]. In [5], the authors further prove that finding the optimal solution for many kinds of congestion games is PLS-Complete, and therefore have initial configurations that require an exponential number of best response rounds to converge to a Nash equilibrium. In seeking to better understand how our games fit into the class of congestion games, we attempted to follow the construction used in [9], but found it both incomprehensible and incorrect. The final section of the paper rectifies these mistakes and simplifies both the notation and the construction used to prove the result.

2 Definitions and Models

2.1 Terms

A *game* has a set N of n players where $n \geq 2$. Each player i has a set S_i consisting of k strategies that i can play. A *state* of the game, or *outcome*, is an $n-$tuple $s = (s_1, s_2, ..., s_n)$ that encodes the strategy used by each player. States can also be notated as (s_i, s_{-i}) which singles out a particular player i using strategy s_i from the rest of the players using strategies s_{-i}. The *utility function* for player i $u_i(s_i, s_{-i})$ takes into account the player's strategy, as well as the strategies used by every other player in the game. A *best response* move is the strategy s_i that maximizes player i's utility function given that every other player is fixed. Drawing on this notion, *best response dynamics* (*BRD*) refers to the process of allowing players to make best response moves in sequence. A *Nash equilibrium* is an outcome where no single player has any incentive to change strategies, or for any player i playing strategy s_i, $u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$ $\forall$ $s'_i \in S_i$. When a Nash Equilibrium is reached via best response dynamics, we say that BRD has converged or terminated. In order to analyze the quality of an outcome, we need an optimal solution. Our optimal solution will always maximize the sum of all player's utilities, so the best outcome is the state s that maximizes $\sum_{i \in N} u_i(s)$, also known as the *Social Welfare*. The *Price of Anarchy* (*PoA*) for a game is the ratio of the Nash equilibrium with the lowest value to the optimal solution. This is frequently used a measure of how bad an outcome could be if best response dynamics were allowed to complete. Similar to the Price of Anarchy, the *Price of Stability* (*PoS*) for a game is the ratio of the Nash equilibrium with the highest value to the optimal solution. A *potential function* $\Phi(s)$ for a game is a function that maps game states to numbers such that, for any two states that differ in only one player's strategy (s_i, s_{-i}) and (s'_i, s_{-i}), the change in the potential function $\Phi(s_i, s_{-i}) - \Phi(s'_i, s_{-i})$ is equal to the change in i's utility $u_i(s_i) - u_i(s'_i)$. The networks used by games will always be a *graph* $G = (V, E)$ consists of a set of vertices V and a set of edges E, where each edge connects exactly two vertices. We will assume unless otherwise specified that there is at most one edge between any pair of vertices and no edges from a vertex to itself.

2.2 Models

Games

In the *Matching Game* there are n players $N = \{1, 2, ..., n\}$ and k strategies $S = \{1, 2, ..., k\}$. Each player shares the same set of strategies and has the same utility function. Let δ^x be the number players using strategy x, then for every player u we define $u(i, s_{-u}) = v_i \delta^i$, or the number of players using a strategy times the value of that strategy. Without loss of generality we always order the strategies such that $0 \leq v_1 \leq v_2 \leq ... \leq v_k$. The *Mismatching Game* uses a very similar setup with n players and k strategies. The only difference lies in the utility function $u(i, s_{-u}) = v_i \sum_{j \neq i} \delta^j$, or the number of players not using strategy i times the value of that strategy.

Superimposing a Network
Adding a network to a game M means adding to the model a graph $G = (V, E)$ where $|V| = n$. We notate this $M \oplus G$. Each vertex in the graph represents a player and each edge corresponds to an instance of the game M. The strategies in $M \oplus G$ are the same as the strategies in G and for a player u, if $\delta(u)$ is the set of nodes adjacent to u and $v_u(s)$ is u's utility function in M, then the utility of u in $M \oplus G$ is $u_u(s) = \sum_{v \in \delta} v_u(s_u, s_v)$ where u and v play strategies s_u and s_v respectively. From here on it is assumed that every game is played on some underlying graph G.

The *Affiliation Game* is only played on networks and is an attempt to merge the Matching and Mismatching games. Given a set of n vertices, we require that the graph G on those vertices be complete and consist of either positive or negative edges. Positive edges correspond to instances of the Matching Game, while negative edges are instances of the Mismatching Game. The result is similar to the Party Affiliation game with an arbitrary number k of distinctly weighted strategies $\{v_1 ... v_k\}$ where $v_1 \leq v_2 \leq ... \leq v_k$. Even though players may engage in two different kinds of games, we assume that the underlying Matching and Mismatching games share the same set of strategies, so each vertex is still playing exactly one strategy in each game.

3 The Matching Game

All of the following results apply to the Matching Game on networks, as described above.

3.1 Convergence

Lemma 1. *When G is a path or cycle, Best Response Dynamics converges in at most $k-1$ rounds. Furthermore, this bound is tight.*

Proof. Note that if a player switches to a strategy as a best response, then they must have at least one neighbor already playing that strategy, or else there would be no reason to switch. Because of this, and because each player has at most two neighbors, once a player has chosen some strategy a as a best response, it will never be a best response for that player to play any strategy b where $v_b \leq v_a$. Consider what would happen if this were not the case. Then let i be the first player who switches to a strategy of a lower value after making at least one other best response move. When i first played strategy a, i had at least one neighbor playing a, so if i's best response is now b, i's neighbor must have changed strategies and because i is the first player to switch to a lower valued strategy, i's neighbor must have switched to a strategy of greater value. Therefore i has one neighbor playing a strategy c where $v_c > v_a \geq v_b$, meaning that i would get more utility playing c than b, so b cannot be i's best response. Therefore no player can switch strategies more than $k-1$ times.

To construct a tight example, consider the case where $k = n$ and each vertex in the path of length k plays the strategy that corresponds to their position in that path. If players take turns from right to left, then the leftmost player will eventually play every strategy in the game, requiring $k-1$ rounds. $\square$

Lemma 2. *When G is complete, BRD converges in one round.*

Proof. Note that because the graph is complete, every player has the same best response move. Therefore the strategy chosen by the first player who moves will be the same strategy chosen by every other player. Similar logic is at work in Lemma 3. □

3.2 Anarchy

Lemma 3. *When G is a complete graph, the Price of Anarchy is* $\frac{v_k}{v_1}$

Proof. It is sufficient to show that any stable solution requires every player to use the same strategy. Consider an instance in which two vertices p and q are playing different strategies. Because the graph is complete, both p and q share all of their other neighbors. Without loss of generality, assume that p is playing strategy i and getting at least as much utility as q. Then q has an incentive to switch because q would receive the same utility as p plus v_i. Because no assumptions were made about p or q, this argument holds for every mismatched pair of players. □

4 The Mismatching Game

4.1 Optimality

Theorem 4. *Finding the optimal solution to an instance of the Networked Mismatch Game (NMG) is NP-Hard, even when* $k = 2$.

Proof. We can prove this by showing that a special case of NMG is sufficient to solve the graph coloring problem. In particular, given a graph coloring instance on a graph G with k colors, we let each color represent a unique strategy in NMG and let all strategies here have the same value. Therefore, the optimal solution to NMG represents the coloring of G where as few neighbors as possible share the same color. If this were not the case (i.e. if there were some coloring that had fewer same-colored adjacent neighbors) then the solution would not be optimal for NMG because greater total utility would have been available using that coloring. Hence, the optimal solution represents the best possible coloring of G with k colors. Note that the chromatic number of a graph is never more than the maximum degree of the graph plus one. Then, by varying the number of strategies, k using a binary search technique, this reduction can be used to find the chromatic number of G.

Furthermore, if $k = 2$ and both strategies have the same value, the optimal solution to NMG on G is also the solution to the max cut problem. Because there are only two strategies, the optimal solution will maximize the number of edges $e = (v_1, v_2)$ where v_1 and v_2 are vertices playing different strategies. This can be used to create the max cut by selecting the vertices playing each strategy as subsets of G - the edges crossing this cut represent the total utility of the solution. Because utility was maximized, so was the number of edges crossing the cut, hence this has solved the max cut problem. □

4.2 Convergence

Theorem 5. *Mismatch games on networks have an exact potential function*

Let $\delta(a)$ be the set of neighbors of a vertex a. Then given a strategy profile (s_a, s_{-a}) the potential for this state is:

$$\Phi(s_a, s_{-a}) \quad = \quad \sum_{b \in \delta(i)} \begin{cases} v_a & \text{if } s_b = s_a \\ v_b + v_a & \text{otherwise.} \end{cases}$$

Proof.
To prove that this is correct, there are three cases. For each case, we will assume that some player a is switching from strategy x to strategy y and that his neighbor b is playing strategy z. Note that when $x = y = z$ this is trivial because nothing has changed.

Case 1: $x \neq z$, $y \neq z$
Before the change, a is getting v_x after the change, a is getting v_y. The difference in a's utility is then $v_x - v_y$, but the change in Φ is $v_x + v_z - (v_y + v_z) = v_x - v_y$.
Case 2: $x = z$, $y \neq z$
The change in a's utility is $-v_y$ and the change in Φ is $v_z - (v_z + v_y) = -v_y$.
Case 3: $x \neq z$, $y = z$
The change in a'a utility is v_x and the change in Φ is $v_x + v_z - v_z = v_x$.

Because the change in Φ mirrors the change in a's utility on each edge whenever a changes strategies, Φ accurately describes the change in potential for any deviating player on any graph. □

Lemma 6. *Best Response Dynamics converges in at most n steps when G is a complete graph*

Proof. Note that because the graph is complete, for a given configuration, each player has the same best response move. Then let the first player p who moves choose strategy i. By definition, p cannot get any more utility at the moment. Furthermore, p's utility is only affected negatively when some other player switches to strategy i - but when that happens, because each player has the same best response move, it is still p's best response to play i. The same argument holds for any player, any strategy, and any timeframe. Therefore, because a player never has an incentive to deviate from the first best response move they make, each player will move at most once before the game reaches an equilibrium. □

Lemma 7. *BRD converges in at most 2 rounds when G is a cycle or a path*

Proof. Note that because each vertex has degree at most two, there are effectively only two cases. In the first case, either $k = 2$ or $v_k > 2v_{k-2}$ meaning that only two strategies will ever be used. The second case is the converse of the first, $k \geq 3$ and $v_k < 2v_{k-2}$ so the top three strategies are all viable. We say a player is *stable* when their best response can never change.
Case 1: Let each player make one best response move in any order, by hypothesis each player is now playing either strategy k or $k-1$. Now let each player take another turn. Consider a player, p, who switched from strategy k to $k-1$. If that was the case, then because $v_k \geq v_{k-1}$, both of p's neighbors must have been playing k. Now, because both of p's neighbors are playing k and have at least one neighbor playing $k-1$, both neighbors are stable, as is p. This means that after a single round of best response moves, the only move that can be made is from $k-1$ to k, and that

can happen only once. Therefore the game has reached a Nash after at most two rounds of BRD.
Case 2: Again, let each player make a single best response move so that every player is playing k, $k-1$ or $k-2$, and again allow each player to make a second move. The table below summarizes the possibilities for the second round.

Neighbors Play	Best Response
$k-2$,$k-2$	k
$k-2$,$k-1$	k
$k-2$,k	$k-1$
$k-1$,$k-1$	k
$k-1$,k	$k-2$
k,k	$k-1$

Note that if a player's best response is k, then neither of their neighbors are currently playing k and furthermore, k will never be a best response for either neighbor. Therefore anyone playing k is stable. Now consider a player p whose best response was $k-1$ and note that because $k-1$ was a best response, p must have at least one neighbor playing k. If p's best response were to become k, then p's neighbor who was playing k must have changed strategies, which is impossible. Furthermore if p's best response were to become $k-2$, then because we know that any neighbors playing k would not have switched, p must have had a neighbor q who switched from $k-2$ to $k-1$, but q's best response could never be $k-1$ because q is a neighbor to p and p is already playing $k-1$. Therefore, in the second round of BRD, only players whose first best response was $k-2$ will ever move, and having moved once, they will be stable. Hence BRD will converge in at most two rounds. $\square$

4.3 Anarchy and Stability

Theorem 8. *The Price of Anarchy for any set of values on any graph is less than or equal to* $\dfrac{(v_k + v_{k-1})\sum_i \frac{1}{v_i}}{2(k-1)}$

Proof. Let $\delta^i(u)$ be the number of neighbors of some vertex u playing a strategy i and $\delta(u)$ be the degree of u. Then if u plays strategy i, u receives a payoff $p_u = v_i(\delta(u) - \delta^i(u))$. For any Nash equilibrium, $p_u \geq v_i(\delta(u) - \delta^i(u))\ \forall i$ because the actual payoff received by any player must be at least as large as the payoff the player would receive by changing strategies. Dividing both sides of this inequality by v_i and taking the sum over all strategies i, $\sum_i \frac{p_u}{v_i} \geq \sum_i(\delta(u) - \delta^i(u))$ or $p_u \sum_i \frac{1}{v_i} \geq k\delta(u) - \delta(u)$ so $p_u \geq \frac{(k-1)\delta(u)}{\sum_i \frac{1}{v_i}}$ for any player u in equilibrium. Taking the sum over all players to obtain a lower bound on the social welfare, $\sum_u p_u \geq \frac{(k-1)}{\sum_i \frac{1}{v_i}} \sum_u \delta(u) = \frac{2m(k-1)}{\sum_i \frac{1}{v_i}}$ where m is the number of edges in the graph.

Because there is a non-zero payoff on any edge only when the strategies played by adjacent players are distinct, the maximum value attainable on any edge given that the strategies have been ordered such that $v_1 \leq v_2 \leq ... \leq v_k$ is $v_k + v_{k-1}$. The value can be obtained on every edge if and only if G is bipartite. So for a general graph G, the value of the optimal solution (OPT) is at most $m(v_k + v_{k-1})$. This implies that the Price of Anarchy, $PoA = \frac{\text{best NE}}{OPT} \leq \frac{(v_k+v_{k-1})\sum_i \frac{1}{v_i}}{2(k-1)}$.

□

Theorem 9. *This bound is tight if and only if* $k-2 < v_1 \sum_{i\neq 1} \frac{1}{v_i}$ *and when the bound is tight, the percentage of players playing each strategy* q *is given by* $\beta_q = \dfrac{\displaystyle\sum_{i\neq q}\prod_{j\neq i} v_j - (k-2)\prod_{i\neq q} v_i}{\displaystyle\sum_{i}\prod_{j\neq i} v_j}$.

Proof. (⇐)
Given that $k-2 < v_1 \sum_{i\neq 1} \frac{1}{v_i}$, it will be useful to transform this inequality before proceeding further. Multiplying the right side by $\frac{\prod_j v_j}{\prod_j v_j}$ gives us (1).

$$k-2 \quad < \quad \frac{\displaystyle\sum_{i\neq 1}\prod_{j\neq i} v_j}{\displaystyle\prod_{j\neq 1} v_j} \tag{1}$$

Now let n be total number of players on one half of a complete bipartite graph, so that every vertex has degree n. Also let n_i be the number of players on one side of the graph playing strategy i. Note that it does not matter which side of the graph we consider, because if there exists an equilibrium for a complete bipartite graph, there exists a symmetric equilibrium, and we will assume without loss of generality that this is the case. At equilibrium, in the worst case, each player receives the same payoff with each strategy so $v_i(n-n_i) = v_j(n-n_j)$ for all pairs of strategies i and j. Now scale n to be 1 and let β_i be the percentage of n that is playing strategy i. These βs must have the following properties:

i. $\sum_i \beta_i = 1$

ii. $\beta_i = 1 - \frac{v_j}{v_i}(1-\beta_j)$

Note that i. is necessary for β_i to be a percentage and ii. follows from the requirement that the payoffs for each strategy be equal.

Lemma 10. $\beta_q = \dfrac{\displaystyle\sum_{i\neq q}\prod_{j\neq i} v_j - (k-2)\prod_{i\neq q} v_i}{\displaystyle\sum_{i}\prod_{j\neq i} v_j}$

Proof. (i.)
Note that each β has a common denominator, so $\sum_q \beta_q = 1$ if and only if $\sum_i \prod_{j\neq i} v_j = \sum_q \sum_{i\neq q} \prod_{j\neq i} v_j - (k-2)\prod_{i\neq q} v_i$. The left side of this equation has each combination of $k-1$ v_i terms represented exactly once. The right side has $k-2$ terms of every value except one and one term of every other $k-1$ combination. There are k terms in the summation, which means that each combination of $k-1$ values appears exactly $k-1$ times positively and $k-2$ times negatively, canceling to equal the left side exactly. □

Proof. (ii.)

$$
\begin{aligned}
\beta_q &= 1 - \frac{v_w}{v_q}(1-\beta_w) && \text{By hypothesis} \\
&= 1 - \frac{v_w}{v_q}\Big(1 - \frac{\sum_{i\neq w}\prod_{j\neq i} v_j - (k-2)\prod_{i\neq w} v_i}{\sum_i \prod_{j\neq i} v_j}\Big) && \text{By definition of } \beta_w \\
&= 1 - \frac{v_w}{v_q}\Big(\frac{\sum_i\prod_{j\neq i} v_j - \sum_{i\neq w}\prod_{j\neq i} v_j - (k-2)\prod_{i\neq w} v_i}{\sum_i \prod_{j\neq i} v_j}\Big) && \text{Use a common denominator} \\
&= 1 - \frac{v_w}{v_q}\Big(\frac{-(k-1)\prod_{i\neq w} v_i}{\sum_i \prod_{j\neq i} v_j}\Big) && \text{Cancel Terms} \\
&= 1 - \frac{(k-1)\prod_{i\neq q} v_i}{\sum_i \prod_{j\neq i} v_j} && \text{Multiply through} \\
&= \frac{\sum_i\prod_{j\neq i} v_j - (k-1)\prod_{i\neq q} v_i}{\sum_i \prod_{j\neq i} v_j} && \text{Use a common denominator} \\
&= \frac{\sum_{i\neq q}\prod_{j\neq i} v_j - (k-2)\prod_{i\neq q} v_i}{\sum_i \prod_{j\neq i} v_j} && \text{Cancel a common term}
\end{aligned}
$$

□

Furthermore, note that (1) implies $(k-2)\prod_{i\neq q} v_i < \sum_{i\neq q}\prod_{j\neq i} v_j$ because $v_1 \leq v_2 \leq ... \leq v_k$ so β_i is positive exactly when (1) holds, which means that the bound on the Price of Anarchy is tight when (1) holds. When (1) does not hold, it means that strategy 1 will never be used by any player on any graph so the Price of Anarchy is tight to the bound in Theorem 1 for the smallest i such that $\{v_i ... v_k\}$ satisfies (1).

□

Proof. ($\Rightarrow$)
Let n_i be the number of players playing strategy i. If the bound on the Price of Anarchy is tight, then because the bound incorporates each of the values $v_1 ... v_k$, a solution which obtains the bound must utilize every strategy. If that were not the case, then one or more strategies could be removed, which would change the bound on the Price of Anarchy but would not change the solution. Therefore $n_i > 0\ \forall i$. For all strategies to be viable in equilibrium, it must be possible for all strategies to obtain parity with every other strategy simultaneously. For example, given three strategies with values 1, 2, and 3, a player v choosing strategy 1 must be able to get as much profit

as someone playing 2 or 3. In order for this to be possible, at least half of v's neighbors must be playing 2 and at least $\frac{2}{3}$ of v's neighbors must be playing 3. This is clearly impossible. In general, for a vertex v of degree $\delta(v)$ playing strategy 1, v's payoff is at most $v_1\delta(v)$, which happens when v's neighbors are all playing strategies other than 1. In that case, it will be profitable for v to stick with strategy 1 as long as $v_1\delta(v) \geq v_i(\delta(v) - \delta^i(v))\ \forall i$. Expanding and dividing each term by $\delta(v)$ shows that $v_1 \geq v_i - v_i\frac{\delta^i(v)}{\delta(v)}$ or $\frac{\delta^i(v)}{\delta(v)} \geq \frac{v_i - v_1}{v_i}$. Summing this inequality over all strategies i, $\frac{1}{\delta(v)}\sum_i \delta^i(v) \geq \sum_i(1 - \frac{v_1}{v_i})$, but because $\delta(v) = \sum_i \delta^i(v)$ this simplifies to $1 \geq k - v_1\sum_i \frac{1}{v_i}$ or $k - 2 \leq v_1 \sum_{i\neq 1}\frac{1}{v_i}$.

Lemma 11. *When $k - 2 = v_1\sum_{i\neq 1}\frac{1}{v_i}$, $\beta_1 = 0$*

Proof. By hypothesis, $v_1 = \dfrac{k-2}{\sum_{i\neq 1}\frac{1}{v_i}}$. So

$$
\begin{aligned}
\beta_1 &= \frac{\sum\limits_{i\neq 1}\prod\limits_{j\neq i} v_j - (k-2)\prod\limits_{i\neq 1} v_i}{\sum\limits_{i}\prod\limits_{j\neq i} v_j} && \text{By Definition} \\
&= \frac{v_1\sum\limits_{i\neq 1}\prod\limits_{j\neq 1,i} v_j - (k-2)\prod\limits_{i\neq 1} v_i}{\sum\limits_{i}\prod\limits_{j\neq i} v_j} && \text{Factor out } v_1 \\
&= \frac{(k-2)\dfrac{\sum\limits_{i\neq 1}\prod\limits_{j\neq 1,i} v_j}{\sum\limits_{i\neq 1}\dfrac{1}{v_i}} - (k-2)\prod\limits_{i\neq 1} v_i}{\sum\limits_{i}\prod\limits_{j\neq i} v_j} && \text{Replace } v_1 \text{ using the equality above} \\
&= \frac{(k-2)\left(\sum\limits_{i\neq 1}\prod\limits_{j\neq 1,i} v_j - \sum\limits_{i\neq 1}\dfrac{1}{v_i}\prod\limits_{i\neq 1} v_i\right)}{\sum\limits_{i\neq 1}\dfrac{1}{v_i}\sum\limits_{i}\prod\limits_{j\neq i} v_j} && \text{Multiply all terms by } \textstyle\sum_{i\neq 1}\frac{1}{v_i} \text{ and factor } (k-2) \\
&= 0 && \text{The terms in the numerator cancel}
\end{aligned}
$$

Because β_1 is zero at equality, and our bound on the Price of Anarchy requires that all strategies be used in the worst case example, if the bound is tight, then the inequality must be strict. □

□

Theorem 12. *When G is complete, $k = 2$, and v_1 is normalized to one, the Price of Anarchy is at most $\dfrac{n^2(v_2+1)^2}{4(v_2 n - 1)(n-1)}$ for even n and $\dfrac{(n-1)(n+1)(v_2+1)^2}{4(v_2 n - 1)(n-1)}$ for odd n.*

Lemma 13. *If G is complete and $k = 2$, the optimal solution to the NMG has value $\frac{n^2(v_1+v_2)}{4}$ for even n and $\frac{(n-1)(n+1)(v_1+v_2)}{4}$ for odd n.*

Proof. Let n_1 be the number of people playing strategy 1 and n_2 be the number of people playing strategy 2. Then note that the value of any solution is exactly $n_1 v_1 n_2 + n_2 v_2 n_1 = n_1 n_2 (v_1 + v_2)$ because this represents the number of people playing strategy 1, each of whom gets v_1 utility from n_2 other players and the number of people playing strategy 2 who each receive v_2 utility from n_1 players. Because $n_1 + n_2 = n$ this can also be written as $n_1(n - n_1)(v_1 + v_2)$, which has n_1 as a single variable. Elementary calculus tells us that the product $n_1(n - n_1)$ is maximized when $n_1 = \frac{n}{2}$. When n is odd, $\frac{n}{2}$ is not integral, so we use the closest integral approximation. The result is then simple multiplication. $n_1 n_2 (v_1 + v_2) = \frac{n}{2}\frac{n}{2}(v_1 + v_2) = \frac{n^2(v_1+v_2)}{4}$. The odd case is the same, letting $n_1 = \frac{n-1}{2}$ and $n_2 = \frac{n+1}{2}$. □

Lemma 14. *If G is complete, $k = 2$, and v_1 is normalized to one, then the worst Nash Equilibrium will have value at least $\frac{(v_2 n+1)(n-1)}{v_2+1}$.*

Proof. As shown above, the optimal solution on this kind of graph splits players as equally as possible between the two strategies. The worst case, therefore, would be the stable solution with the greatest imbalance in the number of players. Because $v_2 \geq v_1$, it will never be stable for more players to play strategy one then strategy two. Therefore we consider the solution where as many players as possible play strategy 2. In order to be a Nash Equilibrium, it must be the case that for any player using strategy 1, $v_1 n_2 \geq v_2(n_1 - 1)$; in other words, strategy 1 is their best response. Substituting $n - n_1$ for n_2 allows us to solve for n_1 yielding $n_1 \geq \frac{v_1(n-1)}{v_1+v_2}$. Because $n_1 + n_2 = n$, this means that $n_2 \leq \frac{v_2 n + v_1}{v_1+v_2}$ with the worst case (in terms of total value) happening when these inequalities are tight. Therefore, if we normalize v_1 to be 1

$$\begin{aligned}
\min(value) &= n_1 n_2 (v_1 + v_2) && \text{Definition of value} \\
&\geq \frac{(n-1)(v_2 n + 1)(v_2 + 1)}{(v_2+1)^2} && \text{Replace terms} \\
&= \frac{v_2 n^2 + v_2^2 n^2 + n - v_2^2 n - v_2 - 1}{(v_2+1)^2} && \text{Expand} \\
&= \frac{(v_2+1)(v_2 n^2 + (1 - v_2)n - 1)}{(v_2+1)^2} && \text{Factor} \\
&= \frac{(v_2 n + 1)(n-1)}{v_2+1} && \text{Factor and simplify}
\end{aligned}$$

□

Theorem 12 follows directly from dividing the optimal result obtained in Lemma 13 by the worst case result obtained in Lemma 14.

Theorem 15. *The Price of Stability for any set of values can be as good as 1 and as bad as the Price of Anarchy, depending on the graph.*

Lemma 16. *When $G = K_n$, $k = 2$, and $nv_1 < v_2$, the Price of Anarchy is the same as the Price of Stability.*

Proof. Recall that K_n is the complete graph on n vertices. Let n_1 be the number of people playing strategy 1 and n_2 be the number of people playing strategy 2. The social welfare in this case is $(n - n_1)(n - n_2)v_1 + (n - n_2)(n - n_1)v_2 = (n - n_1)(n - n_2)(v_1 + v_2)$ which is maximized when $n_1 = n_2 = \frac{n}{2}$. Therefore, the optimal solution will always split players evenly between both strategies for a total value of $\frac{n^2}{4}(v_1 + v_2)$. Note that this depends in no way on the value of the strategies involved.

The only stable solution on K_n for the given values occurs when $n-1$ players play strategy 2 and one player plays strategy 1. Consider a situation in which two players, a and b, and playing strategy 1. Each of them receives a payoff of $(n-2)v_1$. But if a switches to strategy 2, a receives v_2. Since $v_2 > nv_1 > (n-2)v_1$ a has an incentive to change. The same is true whenever more players are playing strategy 1 because that only increases the benefit accrued by switching to strategy 2.

The stable solution described above will have a value of $(n-1)(v_1+v_2)$ meaning that the Price of Stability is $\frac{(n-1)(v_1+v_2)}{\frac{n^2}{4}(v_1+v_2)} = \frac{4(n-1)}{n^2}$. Looking at the limit as n approaches infinity this gives a ratio of $\frac{4}{n}$ or about $\frac{1}{n}$. Note that in this case the Price of Stability is the same as the Price of Anarchy because there is only one Nash Equilibrium.

□

Lemma 17. *When G is bipartite, the Price of Stability is one.*

Proof. For any set of values satisfying (1), the Price of Anarchy on a complete bipartite graph is given by Theorem 8. The optimal solution has one half of the graph playing strategy k and the other half playing strategy $k-1$. Because these are the two best strategies, no person has an incentive to deviate, meaning that the optimal solution is stable and the Price of Stability is 1. □

5 The Affiliation Game

5.1 Potential

Theorem 18. *The Affiliation Game has an exact potential function*

Proof. Note that both the matching game (see [3]) and the mismatching game have exact potential functions, Φ_+ and Φ_-, defined in terms of the change on each edge. On each edge, the potential function for the general affiliation game uses whichever potential function is applicable for that edge (i.e. Φ_+ on all positive edges). Because the component potential functions accurately describe the change in overall utility created by each edge they are applied to, the sum over all of these changes accurately reflects the total change in utility of the affiliation game. □

5.2 Anarchy and Stability

In any graph $G(V, E)$ let m_- be the number of negative edges in G and m_+ be the number of positive edges in G so that $m_- + m_+ = m = |E|$.

Lemma 19. *Given a set of values $\{v_1 ... v_k\}$, the maximum value for the Price of Anarchy in any graph is bounded above by* $\frac{(2v_k m_+ + (v_k+v_{k-1})m_-)\sum_i \frac{1}{v_i}}{2(k-1)m_- + 2m_+}$

Proof. This proof follows exactly the same steps as the proofs on the upper bounds on the Price of Anarchy for the Matching and Mismatching games. □

Let $max_G(PoA)$ be the maximum value for the Price of Anarchy on any graph with a fixed set of values in the Affiliation Game. The corollary below follows directly by trying to maximize the ratio in Lemma 19, which results in m_- becoming zero. Unfortunately, this bound is not tight, as shown in Lemma 21.

Corollary 20. $max_G(PoA) < v_k \sum_i \frac{1}{v_i}$

Lemma 21. *When $m_- = 0$, the Price of Anarchy is $\frac{v_k}{v_1}$*

Proof. Because there are no negative edges, this is really an instance of the matching game on a complete graph. In the matching game, the optimal solution has every player playing v_k. It is then sufficient to show that no stable solution can use more than one strategy. Consider some configuration that uses two strategies, i and j played by groups of players A and B, respectively. Note that if players in A get a better payoff than players in B then every player in B has an incentive to play i. Similarly, if players in B get more payoff, then every player in A has an incentive to deviate to j. Finally, if A and B both receive equal payoffs, then every player has an incentive to deviate because they would receive an additional v_i or v_j utility, respectively. Therefore stable solutions will use only a single strategy. Finally, it is clear that once every player is using the same strategy, no player has an incentive to deviate, because deviating yields no payoff. Therefore in the worst equilibrium, every player plays strategy 1, because it has the lowest value. The desired ratio follows directly from the definition of Price of Anarchy. □

For the following two lemmas, let G be a graph with $2n$ vertices. Vertices are arranged in pairs (groups) connected by a positive edge. Each set of pairs has negative edges to every other vertex. Then let x and y be strategies where $x \geq y$.

Lemma 22. *In any optimal solution to the affiliation game on this graph, positively connected pairs play the same strategy.*

Proof. Let a and b be a mismatching pair. Without loss of generality, let a play x and b be play y. a's contribution to the total value of the solution is currently $(x+y)n_y$ where n_y is the number of players playing y because each edge between a and a vertex playing y yields $x+y$ utility between the two vertices involved. If a were to switch and play y instead, a would provide $(x+y)n_x + 2y$ to the social welfare. Symmetrically b can either contribute $(x+y)n_x$ or $(x+y)n_y + 2x$. Because it must true that either $n_x \geq n_y$ or $n_y \geq n_x$, one player can always be switched while increasing the total social welfare. The same argument holds for any mismatching positively connected pair. □

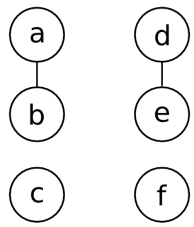
Lemma 23. *The optimal solution to the affiliation game on this graph divides paired groups equally between strategies.*

Proof. Because of the above lemma, it is sufficient to show that deviating the number of groups from an equal distribution yields a lower total value for the solution. Let n_x be the total number of groups playing x and n_y be the total number of groups playing y. Note that each group playing x in contributes $2x + 4xn_y$ to the social welfare because each member of the group receives x for matching their partner and $2xn_y$ for mismatching each group playing y. Similarly, each y group contributes $2y + 4yn_x$. The total value of the proposed optimal solution is therefore $\frac{n}{2}(2y + 4y\frac{n}{2}) + \frac{n}{2}(2x + 4x\frac{n}{2}) = yn + yn^2 + xn + xn^2$.

Now let $\frac{n+m}{2}$ groups play y for some $-n < m < n$. This new solution has a value of $y(n+m) + y(n+m)(n-m) + x(n-m) + x(n-m)(n+m) = yn + ym + yn^2 - ym^2 + xn + xm + xn^2 - xm^2$. Because $m^2 \geq m$ for all m, this new solution has no better value than the original (where $m = 0$). $\square$

Theorem 24. *The Price of Stability on a general graph is not one*

Proof. This proof will use the graph below. Each edge shown is a positive edge, the absence of an edge implies a negative edge.



We will use two strategies called 1 and 2, with values 1 and 2, respectively. Note that the graph is similar to but not exactly the same as the graph for lemmas 22 and 23. It is then sufficient to show that the optimal solution is not the same as the best stable solution. Solutions are represented as matrices with six entries corresponding to the values played by each of the six vertices, $s = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$. By an argument similar to that in Lemma 22, a and b and d and e will play the same strategy in any optimal solution. This fact and symmetry narrow down the viable options to seven, shown below.

$$s_1 = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{bmatrix} \quad s_2 = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} \quad s_3 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \\ 1 & 1 \end{bmatrix} \quad s_4 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \\ 1 & 2 \end{bmatrix} \quad s_5 = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 2 & 2 \end{bmatrix} \quad s_6 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 2 & 2 \end{bmatrix} \quad s_7 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$$

Let $V(s_i)$ be the social value of solution s_i. The table below summarizes the results.

i	1	2	3	4	5	6	7
$V(s_i)$	33	30	32	23	30	28	29

s_1 is not stable, however, because c can obtain a utility of four instead of three by playing strategy 2. Verifying that s_3 is stable is simple - neither a nor c has an incentive to deviate, and symmetry does the rest. Therefore, the Price of Stability on this graph with this strategy set is $\frac{33}{32}$, which is not one.

$\square$

5.3 Observational Results

The following observations were made using a program to simulate the effects of best response dynamics over all possible graphs and a small set of strategies. The project is broken into four files. Graphmaker takes a number of nodes and a number of positive weight (matching) edges, and uses that to construct all possible complete graphs with that many vertices and positive edges. Graphsolver takes a filename prefix and a number of files that contain graphs and finds all nash equilibria and the optimal solution by testing every possible starting configuration and allowing best response dynamics to converge, testing the solution for optimality along the way. Run.sh is a script that, given a target number of players, runs graphmaker and graphsolver for every possible graph and edge distribution up to the target number of players. Combs is a simple program that outputs some combinatorics calculations, enabling run.sh to figure out how many graphs graphmaker produced. The source code has been included as a separate document.

i. The Price of Anarchy is substantially lower than the predicted bound given by Lemma 19.

ii. The Price of Anarchy decreases as the graph goes from being all matching edges to all mismatching edges.

iii. The Price of Stability is very low (lower than 1.1 across all test cases).

iv. Best Response Dynamics converges quickly on average (less than n) and moderately quickly in the worst case, though the relationship between n and the worst case is unclear.

v. Best Response Dynamics converges faster as the graph goes from being all matching edges to all mismatching edges.

6 Potential Games

This proof originally appeared in [9] in an almost incomprehensible form. The proof below uses a heavily refined version of their construction.

Theorem 25. *Every exact potential game is isomorphic to a congestion game.*

Proof.
Formally, a congestion game C as defined in [9] has a set of players $N = \{1, 2, ..., n\}$, a set of facilities $M = \{1, 2, ..., \ell\}$, and a set of strategies Σ^i for every player i where each strategy is a non-empty set of facilities. Further, let v^i be the utility function for a player i in the congestion game. Then for each facility $j \in M$, let $c_j(r)$ be the cost incurred for r people using facility j. If player i plays a strategy consisting of a set of facilities A, $v^i = \sum_{j\in A} c_j(r_j)$ where r_j is the number of players actually using facility j. As seems intuitive, i's utility is exactly the total cost of each facility used by i.

There is, however, an alternative way of expressing the same idea - in particular, one can break down the costs into groups based on the number of people using each facility so that the total cost to i is the sum of the number facilities that only i is using times the cost for one player to use those facilities plus the number of facilities that i and one other player are using times the cost for two players to use those facilities, etc.. To represent this algebraically, we need a way of expressing the set of facilities played by player i and others.

For any subset of players Y, let Y^c denote the complement of Y and $R(Y)$ be the facilities used by the players in Y. Finally, let $R(Y)^c$ be the facilities not used by the players in Y. Then, using i and $\{i\}$ interchangeably, the facilities played by only player i are $R(i) \cap R(i^c)^c$. Similarly, the facilities used by everyone are $R(1) \cap R(2) \cap ... \cap R(n)$. There are other terms that make up v^i, but these two are sufficient for our purposes, so we get equation (1).

$$v^i = \sum_{j \in (R(i) \cap R(i^c)^c)} c_j(1) + ... + \sum_{j \in (\bigcap_{h \in N} R(h))} c_j(n) \quad (2)$$

Given a potential game, let $N = \{1, 2, ..., n\}$ be the set of players (as in the congestion game). Recall that a state m in the potential game can be represented as a vector of length n, that is one entry for each player, so $m = (m_1, m_2, ...m_n)$ where m_i is the strategy that player i is using. Using these definitions we will create an isomorphic congestion game.

In the congestion game there are two kinds of facilities, there is one facility $\alpha(i, m)$ for every player i and every state m in the potential game and one facility $\beta(m)$ for every state m. Then if player i plays strategy m_i in the potential game, they play the unique facilities $\beta(m)$and $\alpha(i, m)$. Conversely, the strategies for player i in the congestion game are exactly the pairs of facilities that share a common state m. From these definitions, it is clear that $\alpha(i, m)$ is a facility that only i is going to play when the potential game is in state m, and $\beta(m)$ is a facility that every player will always play. This leads to equations (2) and (3). To completely squelch the possibility of interference by small groups of players playing the same facilities, we set $c_j(r)$ to be zero for all $1 < r < n$ and all facilities $j \in M$. Now we wish to define $c_j(1)$ and $c_j(n)$ such that the utility of a player in the congestion game is the same as the utility of a player in the potential game.

$$\begin{aligned} |R(i) \cap R(i^c)^c| &= 1 \quad (3) \\ |\textstyle\bigcap_{h \in N} R(h)| &= 1 \quad (4) \end{aligned}$$

This is relatively simple given our notation, we simply let Φ be a potential function for the potential game and then let $c_{\beta(m)}(n) = \Phi(m)$. In order to define $c_{\alpha(i,m)}(1)$, we must first note that the difference between the utility of a player and the potential of state does not depend on the strategy the player is using because the difference in potential between two states that differ in only one player's strategy is exactly the change in utility for that player. Formally, if a player i is playing strategy m_a and switches to m_b we have that $u^i(m_a, m_{-i}) - \Phi(m_a, m_{-i}) = u^i(m_b, m_{-i}) - \Phi(m_b, m_{-i})$. Hence, we define $c_{\alpha(i,m)}(1) = u^i(m) - \Phi(m)$. Therefore the utility of any player i is

$$\begin{aligned} v^i(m) &= \sum_{j \in (R(i) \cap R(i^c)^c)} c_j(1) + ... + \sum_{j \in (\bigcap_{h \in N} R(h))} c_j(n) && \text{By (1)} \\ &= c_{\beta(m)}(n) * 1 + c_{\alpha(i,m)}(1) * 1 && \text{By (2) and (3)} \\ &= u^i(m) - \Phi(m) + \Phi(m) && \text{Def. of } c(1) \text{ and } c(n) \\ &= u^i(m). \end{aligned}$$

$\square$

7 Future Work

There are numerous questions in this realm that remain unanswered. One area that has not been fully explored is the convergence time of best response dynamics in both the matching and mismatching games as function of the underlying graph. Using a case analysis, we have shown here that BRD converges quickly for very simple graphs and preliminary analysis of trees and three-regular graphs indicates that both games should continue to have convergence times that are at least sub-exponential if not constant. However, this analysis becomes increasingly complex as the number of viable strategies increases. Perhaps the better questions to ask consider the potential function. Where are the local maxima for this function? Does the function increase by minimum amount with each best response move? Also, how do any of the games presented here fit into the complexity class PLS? Answering these questions might allow for greater insight into the convergence of BRD on a general graph. Another area in which our knowledge could be expanded is the Price of Anarchy. How does the PoA in the mismatching game depend on the underlying graph? Is there a tight bound for the Price of Anarchy in the affiliation game? If so, under what circumstances is it tight? Finally, there are questions about the Price of Stability. Can we give a better bound on the PoS in the mismatching game and relate that at all to graph structure? Can we find any bound on the PoS in the affiliation game?

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